AD-A099 353 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
SOME OPTIMAL ERROR ESTIMATES FOR PIECEWISE LINEAR FINITE ELEMEN--ETC(U)
MAR 81 R RANNACHER, R SCOTT
UNCLASSIFIED MRC-TSR-2191 NL

END
DATE
FILMED
81
DTIC

LEVEL II



MRC Technical Summary Report #2191

# SOME OPTIMAL ERROR ESTIMATES FOR PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS

Rolf Rannacher and Ridgway Scott

**Mathematics Research Center**
**University of Wisconsin—Madison**
**610 Walnut Street**
**Madison, Wisconsin 53706**

March 1981

(Received January 8, 1981)



**Approved for public release**
**Distribution unlimited**

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

# SOME OPTIMAL ERROR ESTIMATES FOR PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS

Rolf Rannacher* and Ridgway Scott**

Technical Summary Report #2191
March 1981


## ABSTRACT

It is shown that the Ritz projection onto spaces of piecewise linear finite elements is bounded in the Sobolev space, $\overset{\circ}{W}{}^1_p$, for $2 \le p \le \infty$. This implies that for functions in $\overset{\circ}{W}{}^1_p \cap W^2_p$ the error in approximation behaves like $0(h)$ in $W^1_p$, for $2 \le p \le \infty$, and like $0(h^2)$ in $L_p$, for $2 \le p < \infty$. In all these cases the additional logarithmic factor previously included in error estimates for linear finite elements does not occur.


AMS (MOS) Subject Classification: 65N30

Key Words: finite element method, maximum norm

Work Unit Number 3 (Numerical Analysis and Computer Science)

*Institut für Angewandte Mathematik, Universität Erlangen-Nürnberg, 8520 Erlangen, Germany.

**Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

## SIGNIFICANCE AND EXPLANATION

This paper concerns error estimates for methods of approximating the solution of a partial differential equation. The method in question is the so-called "finite element method," which was developed by structural engineers and is now widely used in all branches of engineering. The paper refines previously derived estimates of the error in "maximum norm," i.e. the maximum error (as opposed to an average error). The paper settles certain technical questions as to the rate of convergence of the finite element method in this norm.

| Accession For | |
|---|---|
| NTIS GRA&I | ☑ |
| DTIC TAB | ☐ |
| Unannounced | ☐ |
| Justification | |
| By | |
| Distribution/ | |
| Availability Codes | |
| Dist | Avail and/or Special |
| A | |

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SOME OPTIMAL ERROR ESTIMATES FOR PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS

Rolf Rannacher* and Ridgway Scott**

## 1. Introduction and Results.

Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$, and let $\pi_h = \{K\}$, $0 < h \leq h_0 < 1$, be finite triangulations of $\Omega$ such that the usual regularity condition is satisfied:

(T) The triangles $K \in \pi_h$ only meet in entire common sides or in vertices. Each triangle $K \in \pi_h$ contains a circle of radius $c_1 h$ and is contained in a circle $c_2 h$, where the constants $c_1, c_2$ do not depend on $K$ or $h$.

Corresponding to $\pi_h$, we define the finite dimensional subspace $S_h \subset \mathring{W}^1_\infty$ by

$$S_h = \{v_h \in \mathring{W}^1_\infty : v_h \text{ is linear on each } K \in \pi_h\} ,$$

and the Ritz projection $R_h : \mathring{W}^1_2 \to S_h$ by

$$(\nabla R_h u, \nabla \varphi_h) = (\nabla u, \nabla \varphi_h), \quad \forall \varphi_h \in S_h . \tag{1.1}$$

Here $L_p$ and $W^m_p$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$, are the Lebesgue and Sobolev spaces on $\Omega$ provided with the usual norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. $\mathring{W}^1_p$ in the subspace of those functions in $W^1_p$ which vanish on the boundary in the generalized sense. The inner product of $L_2$ is denoted by $(\cdot,\cdot)$. Finally, by $c$ we mean a generic positive constant which may vary with the context but is always independent of $h$.

Under assumption (T), we have the well known mean-square-error estimates

$$\|u - R_h u\|_{k,2} \leq ch^{2-k}\|u\|_{2,2}, \quad k = 0,1 , \tag{1.2}$$

and the uniform-error estimates (see [4], [8], [6], [1], [7])

$$\|u - R_h u\|_{k,\infty} \leq ch^{2-k}\ln \frac{1}{h}\|u\|_{2,\infty}, \quad k = 0,1 . \tag{1.3}$$

From (1.2) and (1.3) one may conclude by an interpolation argument that for $2 \leq p < \infty$ the $L_p$ error behaves like (see [8])

$$\|u - R_h u\|_p \leq ch^2(\ln \frac{1}{h})^{1-2/p}\|u\|_{2,p} . \tag{1.4}$$

*Institut für Angewandte Matematik, Universität Erlangen-Nürnberg, 8520 Erlangen, Germany

**Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

It has been considered as a challenge among the specialists to remove the additional logarithm in (1.3) and (1.4). This, in particular, since one can show that for higher than second order finite elements these estimates hold without the logarithm (see [5], [8]). Also, for any function $u \in \overset{\circ}{W}{}^1_p \cap W^2_p$, $1 \le p \le \infty$, the natural piecewise linear interpolant $I_h u \in S_h$ is well defined and satisfies

$$(1.5) \qquad \|u - I_h u\|_{k,p} \le ch^{2-k}\|u\|_{2,p}, \quad k = 0,1 .$$

For the case of linear finite elements Fried [2] has recently published an example based on radial symmetry which indicates that (in two and three dimensions) at least the pointwise estimate

$$(1.6) \qquad \|u - R_h u\|_\infty \le ch^2 \ln \frac{1}{h} \|u\|_{2,\infty}$$

may be of optimal order. However, this leaves the question open whether the $L_p$ estimate (1.4) is optimal. In the present paper we shall give an answer to this question for the model situation considered here which is based on the following stability result:

<u>Theorem</u>. <u>Under assumption</u> (T) <u>the Ritz projection</u> $R_h$ <u>is stable in</u> $\overset{\circ}{W}{}^1_p$ <u>for</u> $2 \le p \le \infty$, <u>namely</u>

$$(1.7) \qquad \|R_h u\|_{1,p} \le c\|u\|_{1,p} .$$

The proof of the theorem will be given in the next two sections. One of its consequences is the following

<u>Corollary</u>. <u>Under assumption</u> (T), <u>for any function</u> $u \in \overset{\circ}{W}{}^1_p \cap W^2_p$ <u>there holds</u>

$$(1.8) \qquad \|u - R_h u\|_{1,p} \le ch\|u\|_{2,p}, \quad 2 \le p \le \infty ,$$

$$(1.9) \qquad \|u - R_h u\|_p \le c_p h^2\|u\|_{2,p}, \quad 2 \le p < \infty .$$

<u>Proof</u>. We apply (1.7) for $u - I_h u$ and observe that $R_h \equiv \mathrm{id}$ on $S_h$ to obtain

$$\|R_h u - I_h u\|_{1,p} \le c\|u - I_h u\|_{1,p}, \quad 2 \le p \le \infty .$$

Then, the approximation estimate (1.5) implies (1.8).

To prove (1.9), we use a duality argument. Let $p \in [2,\infty)$, so that $q \equiv p/(p-1) \in (1,2]$. On the convex polygonal domain $\Omega$, the Laplacian is a

homeomorphism from $\overset{\circ}{W}{}^1_q \cap W^2_q$ onto $L_q$, $1 < q \le 2$ (see [3]). Hence there is a $v \in \overset{\circ}{W}{}^1_q \cap W^2_q$ satisfying

$$-\Delta v = \operatorname{sgn}(u - R_h u)\,|u - R_h u|^{p-1} \quad \text{in } \Omega ,$$

and

$$\|v\|_{2,q} \le c\|\Delta v\|_q = c\|u - R_h u\|_p^{p-1} . \tag{1.10}$$

Using now (1.1), Hölder's inequality, (1.5), (1.8), and (1.10), we find

$$\begin{aligned} \|u - R_h u\|_p^p &= (\nabla(u - R_h u),\ \nabla(v - I_h v)) \\ &\le \|u - R_h u\|_{1,p}\|v - I_h v\|_{1,q} \\ &\le \|u - R_h u\|_{1,p}ch\|v\|_{2,q} \\ &\le ch^2\|u\|_{2,p}\|v\|_{2,q} \\ &\le ch^2\|u\|_{2,p}\|u - R_h u\|_p^{p-1} . \end{aligned} \tag{1.11}$$

q.e.d.

We remark on some extensions of our results. The proof of the theorem and to a large extent also that of its corollary make use of the fact that the Laplacian considered as a mapping

$$\Delta : \overset{\circ}{W}{}^1_p \cap W^2_p \to L_p \tag{1.12}$$

is a homeomorphism for $p \in (1,2+\alpha]$, where $\alpha$ is some arbitrarily small but positive number. This is certainly true on a domain with smooth boundary, say $\partial\Omega \in C^2$, for all $\alpha > 0$, and it is known also for convex polygonal domains (see [3]) where $\alpha$ depends on the size of the maximum inner angle, $\omega < \pi$. Our results extend to more general second-order elliptic operators as long as the corresponding mapping (1.12) is a homeomorphism. In the case of a curved boundary the proofs become more involved due to the approximation of $\Omega$ by polygonal domains $\Omega_h$. In the case that $\partial\Omega$ is smooth one can show that for all $p \in (1,\infty]$ the following refined estimate holds:

$$\|R_h u\|_{1,p} \le c\{\|u\|_{1,p;\Omega_h} + h^{1-1/p}\|u\|_{1,p;\Omega\setminus\Omega_h}\} . \tag{1.3}$$

From that estimate one can again draw the conclusions (1.8) and (1.9), now valid for all $p \in (1,\infty]$ and $p \in (1,\infty)$, respectively. The results for $1 < p < 2$ are proved

via a duality argument that makes use of elliptic regularity results that are not generally valid for non-smooth boundaries.

## 2. Proof of the Theorem.

Notation and techniques are similar to those used in [1]. However, the key difference is in the type of Green's function employed. The basic technique used in several papers is to reduce to the problem of estimating the Galerkin error $g - g^h$ in approximating the solution of

$$-\Delta g = \delta \quad \text{in} \quad \Omega$$

where $\delta$ is the Dirac $\delta$-function or some approximation to it. The difficulty is that, with piecewise linear approximation, the error $g - g^h$ contains a logarithmic factor. For example, it was noted in [8] that $0 < c_1 \leq h^{-1}(\ln h^{-1})^{-1}\|g - g^h\|_{W_1^1} \leq c_2$ as $h \to 0$. The reason is that the smoothness of $g$ is such that piecewise linears fail to afford optimal approximation (whereas higher degree piecewise polynomials would yield an approximation rate devoid of the logarithmic factor). The remedy here is to consider instead a "derivative" Green's function, satisfying

$$-\Delta g = \frac{\partial \delta}{\partial x_i} \quad \text{in} \quad \Omega$$

(for each $i = 1,2$). Now $g$ is more singular, and piecewise linears afford optimal approximation, albeit at a slower rate. We now turn to the details.

Let $u \in \overset{\circ}{W}{}_p^1$, $2 \leq p \leq \infty$, be given. We pick any point $z \in \Omega$ contained in the interior of some triangle $K_z \in \pi_h$, and denote by $\partial$ any of the operators $\partial/\partial x_i$, $i = 1,2$. Because of assumption (T) there is a function $\delta_z \in C_0^\infty(K_z)$ such that

$$\int \delta_z dx = 1, \quad |\nabla^k \delta_z| \leq ch^{-2-k}, \quad k = 0,1,\ldots, \tag{2.1}$$

where the constant $c$ does not depend on $z$ or $h$. Then, by construction,

$$\partial\varphi_h(z) = (\partial\varphi_h, \delta_z), \quad \forall\, \varphi_h \in S_h . \tag{2.2}$$

Correspondingly, we define $g_z \in \overset{\circ}{W}{}_2^1$ by

$$(\nabla g_z, \nabla\varphi) = (\delta_z, \partial\varphi), \quad \forall\, \varphi \in \overset{\circ}{W}{}_2^1 . \tag{2.3}$$

Clearly, $g_z$ is a regularized derivative of the Green's function of the Laplacian on $\Omega$. Using this notation, we have

$$\partial R_h u(z) = (\nabla R_h u, \nabla g_z) = (\nabla u, \nabla R_h g_z) \tag{2.4}$$
$$= (\partial u, \delta_z) - (\nabla u, \nabla(g_z - R_h g_z)) .$$

We introduce the weight function

$$\sigma_z(x) = (|x - z|^2 + \kappa^2 h^2)^{1/2}, \quad \kappa \geq 1 \tag{2.5}$$

where the parameter $\kappa$ will be chosen appropriately large, $\kappa \geq \kappa_* \geq 1$, but independent of $h$! We note that from now on the generic constant $c$ is also independent of $\kappa$ and $z \in \Omega$, and of the parameter $\alpha \in (0,1]$ introduced below.

Suppose temporarily that $p < \infty$. Applying Hölder's inequality to the terms in (2.4), we obtain for any $\alpha \in (0,1]$ that

$$|(\nabla u, \nabla(g_z - R_h g_z))| \leq (\int \sigma_z^{-2-\alpha}|\nabla u|^p dx)^{1/p}(\int \sigma_z^{-2-\alpha} dx)^{\frac{p-2}{2p}} (\int \sigma_z^{2+\alpha}|\nabla(g_z - R_h g_z)|^2 dx)^{1/2}$$
$$\leq c(\alpha^{-1}h^{-\alpha})^{\frac{p-2}{2p}} M_h (\int \sigma_z^{-2-\alpha}|\nabla u|^p dx)^{1/p} ,$$

where

$$M_h = \max_{z \in \Omega} (\int \sigma_z^{2+\alpha}|\nabla(g_z - R_h g_z)|^2 dx)^{1/2} .$$

Furthermore,

$$|(\partial u, \delta_z)| \leq (\int_{K_z} |\partial u|^p dx)^{1/p}(\int_{K_z} |\delta_z|^{\frac{p}{p-1}} dx)^{\frac{p-1}{p}}$$
$$\leq ch^{-\frac{2}{p}} (\int_{K_z} |\nabla u|^p dx)^{1/p} .$$

We apply the above estimates with (2.4), raise to the p-th power and integrate with respect to $z \in \Omega$ to obtain

$$\|\partial R_h u\|_p \le c(h^{-2} \iint_{K_z} |\nabla u|^p dx\, dz)^{1/p} +$$

$$+ c(\alpha^{-1}h^{-\alpha})^{\frac{p-2}{2p}} M_h (\iint \sigma_z^{-2-\alpha} |\nabla u|^p dx\, dz)^{1/p} .$$

Thus, by interchanging integration, we find

$$\|\partial R_h u\|_p \le c\|\nabla u\|_p (1 + \alpha^{-1/2} h^{-\alpha/2} M_h) , \tag{2.6}$$

where the constant $c$ is obviously independent of $p$. Estimate (2.6) is also easily seen to hold for $p = \infty$ using the above techniques. Now, to prove the assertion of the theorem, we have to show that

$$M_h = \max_{z \in \Omega} (\int \sigma_z^{2+\alpha} |\nabla(g_z - R_h g_z)|^2 dx)^{1/2} \le c_\alpha h^{\alpha/2} , \tag{2.7}$$

for a proper choice of $\alpha \in (0,1]$.

To prove (2.7), we need some preparations. From now on, we drop the subscript $z$ and simply write $\sigma, g$ for $\sigma_z, g_z$.

The weight function $\sigma$ satisfies

$$|\nabla_k \sigma| \le c\sigma^{1-k} \le c(\kappa h)^{1-k}, \quad k = 0,1,2,\dots . \tag{2.8}$$

*Here* $\nabla_k \sigma$ *denotes the tensor of k-th order derivatives of* $\sigma$. Moreover, for $\kappa \ge \kappa_1$ sufficiently large, one has that (see [5])

$$\max_{K \in \pi_h} [\max_{x \in K} \sigma(x)/\min_{x \in K} \sigma(x)] \le c \tag{2.9}$$

holds uniformly for $z \in \Omega$. For any function $v \in \overset{\circ}{W}{}_2^1 \cap [\prod_{K \in \pi_h} W_2^2(K)]$ the natural piecewise linear interpolant $I_h v \in S_h$ is well defined and satisfies

$$\|\nabla(v - I_h v)\|_{2;K} \le ch\|\nabla_2 v\|_{2;K}, \quad K \in \pi_h . \tag{2.10}$$

Combining (2.10) and (2.9), one easily sees that the following holds:

$$\int \sigma^\beta |\nabla(v - I_h v)|^2 dx \le ch^2 \int' \sigma^\beta |\nabla_2 v|^2 dx , \tag{2.11}$$

where the abbreviation used is

$$\int' \dots dx = \sum_{K \in \pi_h} \int_K \dots dx .$$

To prove (2.7), we set $\psi = \sigma^{2+\alpha}(g - R_h g)$ and we use (1.1) to obtain

$$\int \sigma^{2+\alpha}|\nabla(g - R_h g)|^2 dx = \int \nabla(g - R_h g)\nabla(\psi - I_h\psi)dx + \frac{1}{2}\int \Delta\sigma^{2+\alpha}(g - R_h g)^2 dx .$$

Thus,

$$\int \sigma^{2+\alpha}|\nabla(g - R_h g)|^2 dx \le \int \sigma^{-2-\alpha}|\nabla(\psi - I_h\psi)|^2 dx + c\int \sigma^{\alpha}(g - R_h g)^2 dx . \tag{2.12}$$

From (2.11), we get by a simple calculation that

$$\int \sigma^{-2-\alpha}|\nabla(\psi - I_h\psi)|^2 dx \le ch^2 \int \sigma^{2+\alpha}|\nabla_2|^2 dx +$$
$$+ c\kappa^{-2}\left(\int \sigma^{2+\alpha}|\nabla(g - R_h g)|^2 dx + \int \sigma^{\alpha}(g - R_h g)^2 dx\right) .$$

We insert this estimate into (2.12) and find that, for $\kappa \geq \kappa_2$ sufficiently large,

$$\int \sigma^{2+\alpha}|\nabla(g - R_h g)|^2 dx \le ch^2 \int \sigma^{2+\alpha}|\nabla_2 g|^2 dx + c\int \sigma^{\alpha}(g - R_h g)^2 dx . \tag{2.13}$$

To handle the second term on the right side of (2.13), we employ a duality argument in weighted norms. For fixed $h$, let $v \in \overset{\circ}{W}{}_2^1$ be the solution of the auxiliary problem

$$-\Delta v = \sigma^{\alpha}(g - R_h g) \quad \text{in } \Omega . \tag{2.14}$$

Since $\Omega$ is convex, it is guaranteed that $v \in W_2^2$. Moreover, in Section 3 we shall show that the following weighted a priori estimate holds for all $v \in \overset{\circ}{W}{}_2^1(\Omega)$ such that $\Delta v \in \overset{\circ}{W}{}_2^1(\Omega)$:

$$\int \sigma^{-2-\alpha}|\nabla_2 v|^2 dx \le c\alpha^{-1}(\kappa h)^{-2}\int \sigma^{2-\alpha}|\nabla\Delta v|^2 dx . \tag{2.15}$$

Consequently,

$$\int \sigma^{-2-\alpha}|\nabla_2 v|^2 dx \le c\alpha^{-1}(\kappa h)^{-2}\int \left(\sigma^{2+\alpha}|\nabla(g - R_h g)|^2 + \sigma^{\alpha}(g - R_h g)^2\right)dx . \tag{2.16}$$

Using (2.14), we have

$$\int \sigma^{\alpha}(g - R_h g)^2 dx = \int \nabla(g - R_h g)\cdot\nabla(v - I_h v)dx$$
$$\le \left(\int \sigma^{2+\alpha}|\nabla(g - R_h g)|^2 dx\right)^{1/2}\left(\int \sigma^{-2-\alpha}|\nabla(v - I_h v)|^2 dx\right)^{1/2} .$$

Then, by (2.11) and (2.16), choosing $\kappa \geq \kappa_3$ yields

$$\int \sigma^{\alpha}(g - R_h g)^2 dx \le c(\alpha\kappa)^{-1}\int \sigma^{2+\alpha}|\nabla(g - R_h g)|^2 dx . \tag{2.17}$$

We insert (2.17) into (2.13) and choose again $\kappa \geq \kappa_4(\alpha)$ sufficiently large to obtain

(2.18) $$\int \sigma^{2+\alpha}|\nabla(g_z - R_h g_z)|^2 dx \leq ch^2 \int \sigma^{2+\alpha}|\nabla_2 g|^2 dx \ .$$

Thus, we have reduced the proof of (2.7) to an a priori estimate of the form

(2.19) $$\int \sigma^{2+\alpha}|\nabla_2 g|^2 dx \leq c_\alpha h^{\alpha-2} \ .$$

This estimate, however, is an obvious consequence of the a priori estimate

(2.20) $$\int \sigma^{2+\alpha}|\nabla_2 g|^2 dx \leq c \int \sigma^{2+\alpha}(\partial\delta)^2 dx + c\alpha^{-1}(\kappa h)^{-2} \int \sigma^{2+\alpha}\delta^2 dx \ ,$$

which will be proven in Section 3, for $0 < \alpha \leq \alpha_\Omega$ sufficiently small.

## 3. Some Weighted A Priori Estimates.

Let functions $f \in \overset{\circ}{W}{}_2^1$ and $b \in [\overset{\circ}{W}{}_2^1]^2$ be given, and let $v \in \overset{\circ}{W}{}_2^1$ be such that

(3.1) $$-\Delta v = f + \operatorname{div} b \quad \text{in } \Omega \ .$$

If $\sigma = (|x - z|^2 + \zeta^2)^{1/2}$ is the weight function introduced in Section 2, then we have the following

Lemma. For any convex polygonal domain $\Omega$, there exists an $\alpha_\Omega \in (0,1]$ such that for all parameter values $\alpha \in (0,\alpha_\Omega]$ the following a priori estimates hold,

(i) if $f \equiv 0$:

(3.2) $$\int \sigma^{2+\alpha}|\nabla_2 v|^2 dx \leq c \int \sigma^{2+\alpha}|\operatorname{div} b|^2 dx + c\alpha^{-1}\zeta^{-2} \int \sigma^{2+\alpha}|b|^2 dx \ ,$$

(ii) if $b \equiv 0$:

(3.3) $$\int \sigma^{-2-\alpha}|\nabla_2 v|^2 dx \leq c\alpha^{-1}\zeta^{-2} \int \sigma^{2-\alpha}|\nabla f|^2 dx \ .$$

Proof. (i) To prove (3.2), we estimate

$$\int \sigma^{2+\alpha}|\nabla_2 v|^2 dx \leq \int |\nabla_2[\sigma^{1+\alpha/2}v]|^2 dx + c \int \{\sigma^\alpha|\nabla v|^2 + \sigma^{\alpha-2}v^2\} dx \ .$$

Since $\Omega$ is convex, we have the standard $L_2$ a priori estimate

$$\|w\|_{2,2} \leq c\|\Delta w\|_2, \quad w \in \overset{\circ}{W}{}_2^1 \cap W_2^2 \ .$$

Applying this to $\sigma^{1+\alpha/2}v$, we find by a simple calculation that

$$(3.4)\qquad \int \sigma^{2+\alpha}|\nabla_2 v|^2 dx \le c \int \sigma^{2+\alpha}|\text{div } b|^2 dx + c \int \{\sigma^{\alpha}|\nabla v|^2 + \sigma^{\alpha-2}v^2\} dx \ .$$

Furthermore,

$$\int \sigma^{\alpha}|\nabla v|^2 dx = \int \nabla v \nabla(\sigma^{\alpha} v) dx + \frac{1}{2}\int \Delta\sigma^{\alpha} v^2 dx \ ,$$

and hence, using (3.1),

$$(3.5)\qquad \int \sigma^{\alpha}|\nabla v|^2 dx \le c \int \sigma^{2+\alpha}|\text{div } b|^2 dx + c \int \sigma^{\alpha-2} v^2 dx \ .$$

Combining (3.5) with (3.4), we arrive at

$$(3.6)\qquad \int \sigma^{2+\alpha}|\nabla_2 v|^2 dx \le c \int \sigma^{2+\alpha}|\text{div } b|^2 dx + c \int \sigma^{\alpha-2} v^2 dx \ .$$

Next, we apply Hölder's inequality to obtain

$$(3.7)\qquad \begin{aligned} \int \sigma^{\alpha-2} v^2 dx &\le \left(\int \sigma^{-2-\alpha} dx\right)^{(2-\alpha)/(2+\alpha)} \|v\|^2_{(2+\alpha)/\alpha} \\ &\le c(\alpha^{-1}\zeta^{-\alpha})^{(2-\alpha)/(2+\alpha)} \|v\|^2_{(2+\alpha)/\alpha} \end{aligned}$$

We have already noted that the Laplacian is a homeomorphism from $\overset{\circ}{W}{}^1_q \cap W^2_q$ onto $L_q$ for all $q \in (1,2]$. Hence, there is a $w \in \overset{\circ}{W}{}^1_{(2+\alpha)/2} \cap W^2_{(2+\alpha)/2}$ satisfying

$$-\Delta w = \text{sgn}(v)|v|^{2/\alpha} \quad \text{in} \quad \Omega \ ,$$

and

$$(3.8)\qquad \|w\|_{2,(2+\alpha)/2} \le c\|\Delta w\|_{(2+\alpha)/2} \ .$$

Then, we have via Hölder's inequality, Sobolev's inequality, and (3.8) that

$$\begin{aligned} \|v\|^{(2+\alpha)/\alpha}_{(2+\alpha)/\alpha} = (\nabla v, \nabla w) = (b, \nabla w) &\le \|b\|_{(4+2\alpha)/(2+3\alpha)} \|w\|_{1,(4+2\alpha)/(2-\alpha)} \\ &\le c\|b\|_{(4+2\alpha)/(2+3\alpha)} \|w\|_{2,(2+\alpha)/2} \le c\|b\|_{(4+2\alpha)/(2+3\alpha)} \|v\|^{2/\alpha}_{(2+\alpha)/\alpha} \ . \end{aligned}$$

Thus, we obtain

$$(3.9)\qquad \|v\|_{(2+\alpha)/\alpha} \le c\|b\|_{(4+2\alpha)/(2+3\alpha)} \ .$$

Now, again by Hölder's inequality,

$$(3.10)\qquad \begin{aligned} \|b\|_{(4+2\alpha)/(2+3\alpha)} &\le \left(\int \sigma^{2+\alpha}|b|^2 dx\right)^{1/2} \left(\int \sigma^{-[2+\alpha]^2/2\alpha} dx\right)^{\alpha/(2+\alpha)} \\ &\le c\zeta^{-(4+\alpha^2)/(4+2\alpha)} \left(\int \sigma^{2+\alpha}|b|^2 dx\right)^{1/2} \ . \end{aligned}$$

Combining the estimates (3.10)-(3.7), we obtain that for $\alpha \in (0,1]$

$$\int \sigma^{\alpha-2} v^2 dx \le c\alpha^{-1}\zeta^{-2} \int \sigma^{2+\alpha}|b|^2 dx \,. \tag{3.11}$$

This together with (3.6) proves the estimate (3.2) for the choice $\alpha_\Omega = 1$.

(ii) To prove (3.3), we apply Hölder's inequality as follows:

$$\begin{aligned} \int \sigma^{-2-\alpha}|\nabla_2 v|^2 dx &\le \left(\int \sigma^{-(2+\alpha)/\alpha} dx\right)^{\alpha} \|\nabla_2 v\|^2_{2/(1-\alpha)} \\ &\le c\zeta^{\alpha-2}\|v\|^2_{2,2/(1-\alpha)} \,. \end{aligned} \tag{3.12}$$

Above, we have noted that the Laplacian is a homeomorphism from $\overset{\circ}{W}{}^1_{2/(1-\alpha)} \cap W^2_{2/(1-\alpha)}$ onto $L_{2/(1-\alpha)}$ for $\alpha \in (0,\alpha_\Omega]$, where $1 > \alpha_\Omega > 0$ is determined by the maximum inner angle of $\Omega$. Thus, for $\alpha \in (0,\alpha_\Omega]$, we have that

$$\|v\|_{2,2/(1-\alpha)} \le c\|\Delta v\|_{2/(1-\alpha)} \,. \tag{3.13}$$

By Sobolev's inequality combined with Poincare's inequality (notice that $\Delta v \in \overset{\circ}{W}{}^1_2$),

$$\|\Delta v\|_{2/(1-\alpha)} \le c\|\nabla\Delta v\|_{2/(2-\alpha)} \,. \tag{3.14}$$

We apply again Hölder's inequality to obtain

$$\begin{aligned} \|\nabla\Delta v\|_{2/(2-\alpha)} &\le \left(\int \sigma^{-(2-\alpha)/(1-\alpha)} dx\right)^{(1-\alpha)/2} \left(\int \sigma^{2-\alpha}|\nabla\Delta v|^2 dx\right)^{1/2} \\ &\le c\alpha^{-1/2}\zeta^{-\alpha/2}\left(\int \sigma^{2-\alpha}|\nabla\Delta v|^2 dx\right)^{1/2} \,. \end{aligned} \tag{3.15}$$

Combining the estimates (3.15) - (3.12), we finally reach the desired estimate

$$\int \sigma^{-2-\alpha}|\nabla_2 v|^2 dx \le c\alpha^{-1}\zeta^{-2} \int \sigma^{2-\alpha}|\nabla\Delta v|^2 dx \,,$$

valid for $\alpha \in (0,\alpha_\Omega]$. q.e.d.

## REFERENCES


[1] J. Frehse and R. Rannacher, "Eine $L^1$-Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente," Finite Elemente, Tagungsband Bonn. Math. Schr. vol. 89 (1976), pp. 92-114.

[2] I. Fried, "On the optimality of the pointwise accuracy of the finite element solution," Int. J. Numer. Meth. Eng. vol. 15 (1980), pp. 451-456.

[3] P. Grisvard, "Behavior of solutions of an elliptic boundary value problem in polygonal or polyhedral domains," Numerical Solution of Partial Differential Equations - III (Synspade 1975), B. Hubbard, ed., Academic Press, New York, 1976, pp. 207-274.

[4] F. Natterer, "Über die punktweise Konvergenz finiter Elemente," Numer. Math. vol. 25 (1975), pp. 67-77.

[5] J. A. Nitsche, "$L^\infty$-convergence of finite element approximation," Second Conf. on Finite Elements, Rennes, France (1975).

[6] J. Nitsche, "$L_\infty$-convergence of finite element approximations," Mathematical Aspects of Finite Element Methods, I. Galligani and E. Magenes, eds., Lect. Notes in Math. vol. 606, Springer-Verlag, Berlin, 1977, pp. 261-274.

[7] A. H. Schatz and L. B. Wahlbin, "Maximum norm estimates in the finite element method on plane polygonal domains. Part I," Math. Comp. vol. 32 (1978), pp. 73-109.

[8] R. Scott, "Optimal $L^\infty$ estimates for the finite element method on irregular meshes," Math. Comp. vol. 30 (1976), pp. 681-697.


RR/RS/ed

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>2191 ✓ | 2. GOVT ACCESSION NO.<br>AD-A099 353 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>SOME OPTIMAL ERROR ESTIMATES FOR PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS | | 5. TYPE OF REPORT & PERIOD COVERED<br>Summary Report - no specific reporting period |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>Rolf Rannacher and Ridgway Scott | | 8. CONTRACT OR GRANT NUMBER(s)<br>DAAG29-80-C-0041 ✓ |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Mathematics Research Center, University of Wisconsin<br>610 Walnut Street<br>Madison, Wisconsin 53706 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>Work Unit Number 3 - Numerical Analysis and Computer Science |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>U. S. Army Research Office<br>P.O. Box 12211<br>Research Triangle Park, North Carolina 27709 | | 12. REPORT DATE<br>March 1981 |
| | | 13. NUMBER OF PAGES<br>11 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

finite element method, maximum norm

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)


It is shown that the Ritz projection onto spaces of piecewise linear finite elements is bounded in the Sobolev space, $\overset{\circ}{W}{}^1_p$, for $2 \le p \le \infty$. This implies that for functions in $\overset{\circ}{W}{}^1_p \cap W^2_p$ the error in approximation behaves like $O(h)$ in $W^1_p$, for $2 \le p \le \infty$, and like $O(h^2)$ in $L_p$, for $2 \le p < \infty$. In all these cases the additional logarithmic factor previously included in error estimates for linear finite elements does not occur.


DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DATE
FILMED

8